A disadvantage of both the Window and the Frequency Sampling methods is that the pass band and the stop band edge frequencies are not accurately determined. The equiripple filter is more complicated to design but gives more exact control in the design of the filter.

5 IIR Filter Design

There are two main methods:

1) Directly in the *z*-plane where zeros poles are placed to approximate the required response.

2) Digitising the transfer function of an analogue filter by using a transformation from the continuous to the discrete time domain. The design procedure in this case is as follows:

    a. Determine the filter requirements.

    b. Find a suitable transfer function $H(s)$.

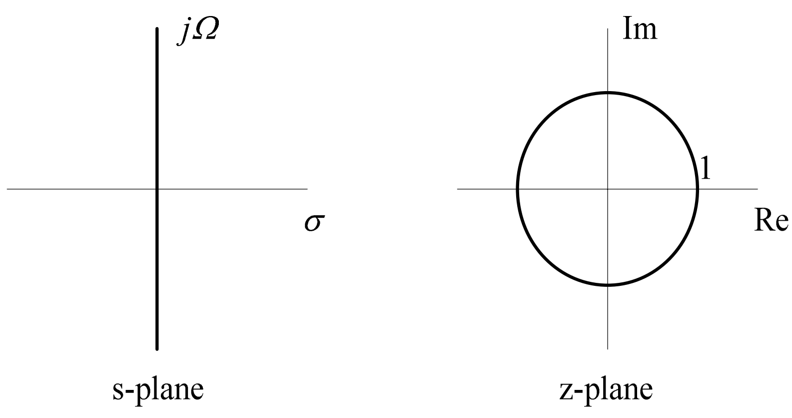
    c. Choose a transformation (mapping) from $H(s)$ to $H(z)$.

    d. Implement $H(z)$.

5.1 Desirable Properties of the Transformation/Mapping

To preserve the frequency selective properties of the continuous time spectrum, we require that:

(1) the $j\Omega$ axis maps to the unit circle. Note that Ω is being used to denote radial frequency in rad/s (continuous time domain), whereas ω will be used to denote radial frequency in rad/sample (discrete time domain). This is shown in Figure



$$H(\Omega) = H(s)\big|_{s=j\Omega} \qquad H(\omega) = H(z)\big|_{z=e^{j\omega}}$$

Figure 6: Mapping of frequencies

(2) Re $\{s\} < 0$ maps to $|z| < 1$. This will ensure that a stable analogue filter will map onto a stable digital one. This is shown in Figure 7.

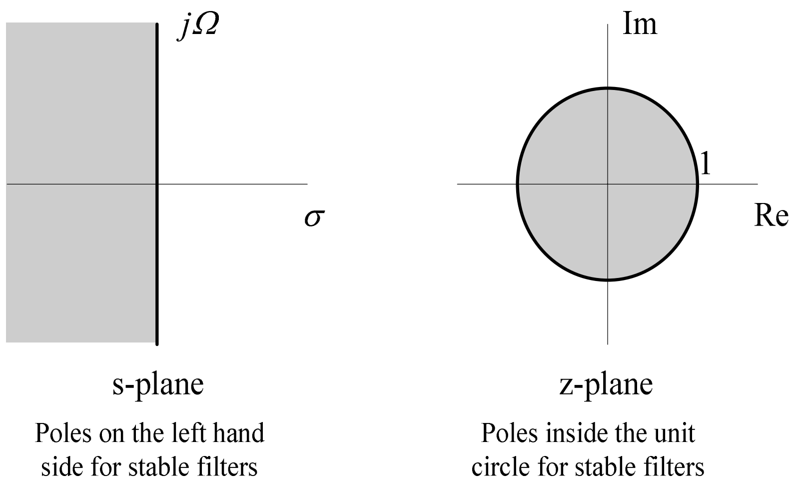


Figure 7: Mapping of stable filters

5.2 Mapping Methods

1) Mapping of differentials (Property (1) not satisfied)

2) Impulse Invariant (Property (1) is only satisfied if Ω is band limited)

3) Bilinear Transformation (Both properties satisfied)

4) Matched z-Transform (Property (1) is only satisfied if Ω is band limited)

5.3 The Bilinear Transformation

H(*s*) is mapped onto *H*(*z*) by using the following transformation:

$$z = \frac{\frac{2}{T_s} + s}{\frac{2}{T_s} - s} \tag{5.1}$$

where T_s is the sampling interval.

From equation (5.1) we may deduce that

$$s = \frac{2}{T_s}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{5.2}$$

To verify that the properties given in Section 5.1 are satisfied, let $s = j\Omega$, then

$$z = \frac{\frac{2}{T_s} + j\Omega}{\frac{2}{T_s} - j\Omega}$$

$$\Rightarrow |z|^2 = \frac{\left(\frac{2}{T_s}\right)^2 + \Omega^2}{\left(\frac{2}{T_s}\right)^2 + \Omega^2} = 1 \qquad (5.3)$$

Therefore the $j\Omega$ axis maps onto $|z| = 1$, i.e. the unit circle. Hence property (1) is satisfied.

Now consider property (2).

If $\sigma < 0$, then

$$\sigma = \mathrm{Re}\left\{\frac{2}{T_s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right\} < 0$$

$$\Rightarrow \mathrm{Re}\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\} < 0$$

Let $z = a + jb$.

$$\mathrm{Re}\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\} = \mathrm{Re}\left\{\frac{z-1}{z+1}\right\}$$

$$= \mathrm{Re}\left\{\frac{a+jb-1}{a+jb+1}\right\} = \frac{(a+1)(a-1)+b^2}{(a+1)^2+b^2}$$

$$= \frac{a^2+b^2-1}{a^2+b^2+2a+1} = \frac{|z|^2-1}{|z|^2+2a+1}$$

Hence, if $\sigma < 0$, then $\frac{|z|^2-1}{|z|^2+2a+1} < 0$

Let $z = re^{j\Omega} = a + jb$. Then $|z| = r$ and $a = r\cos\Omega$.

$$\Rightarrow \frac{r^2-1}{r^2+2r\cos\Omega+1} < 0$$

Therefore, either

$$r^2 - 1 < 0 \text{ and } r^2 + 2r\cos\Omega + 1 > 0 \qquad (5.4)$$

or

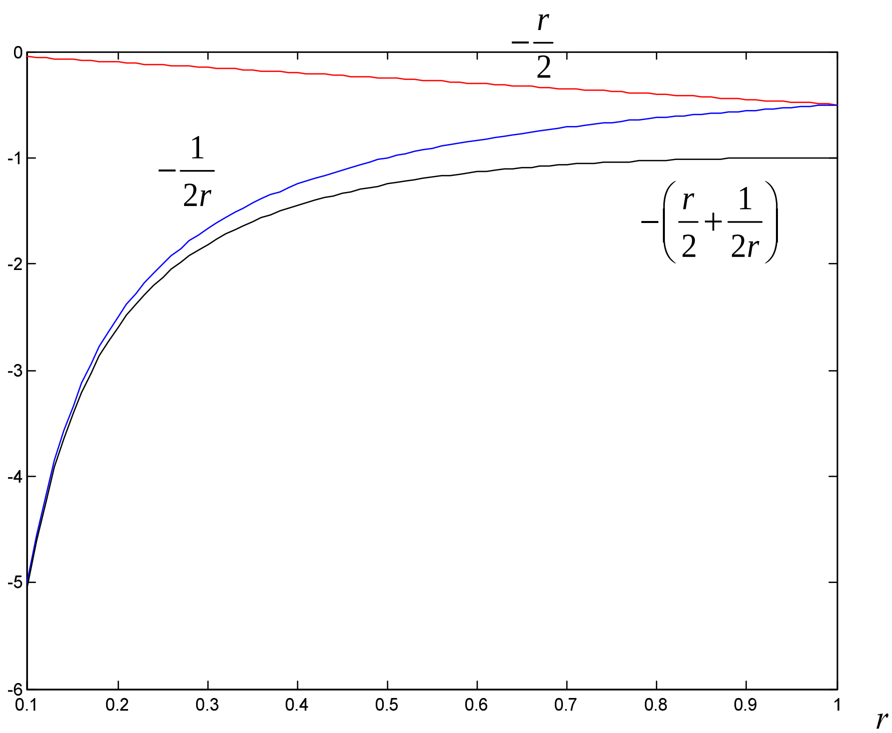
$$r^2 - 1 > 0 \text{ and } r^2 + 2r\cos\Omega + 1 < 0 \tag{5.5}$$

Consider first (5.4). $r^2 - 1 < 0 \quad \Rightarrow \quad r < 1$.

Also, $\qquad r^2 + 2r\cos\Omega + 1 > 0$

$$\Rightarrow \cos\Omega > \frac{-(r^2+1)}{2r} = -\left(\frac{r}{2} + \frac{1}{2r}\right)$$

But $0 < r < 1$.



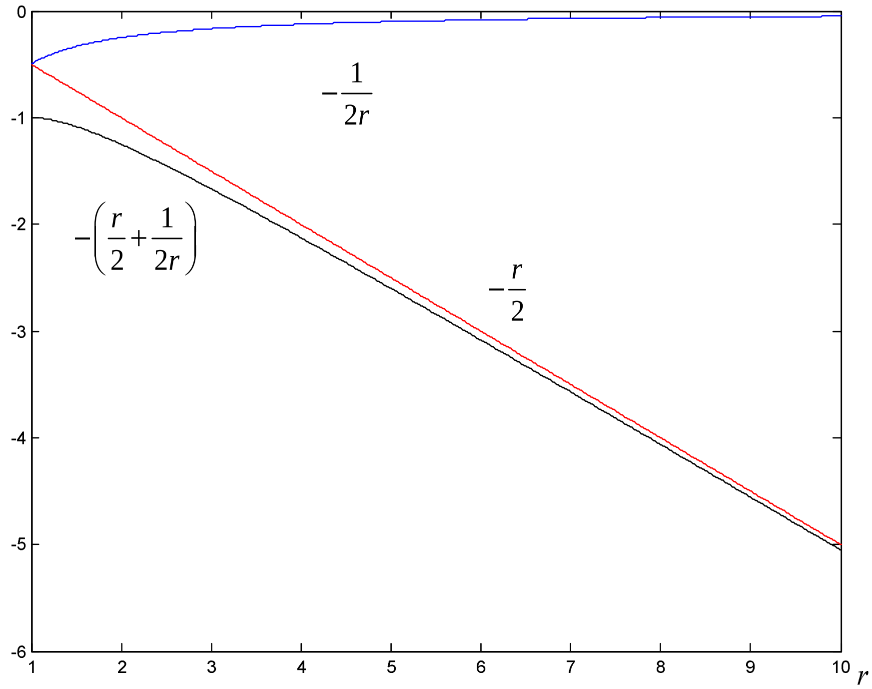
Therefore $-\left(\frac{r}{2} + \frac{1}{2r}\right) < -1 \quad \Rightarrow \quad \cos\Omega > -1$. We may find some Ω that satisfies $\cos\Omega > -1$ giving us $r < 1$ (or $|z|$ as required).

Consider next the case given by (5.5). $r^2 - 1 > 0 \quad \Rightarrow \quad r > 1$.

Also, $\qquad r^2 + 2r\cos\Omega + 1 < 0$

$$\Rightarrow \cos\Omega < \frac{-(r^2+1)}{2r} = -\left(\frac{r}{2} + \frac{1}{2r}\right)$$

But $r > 1$.



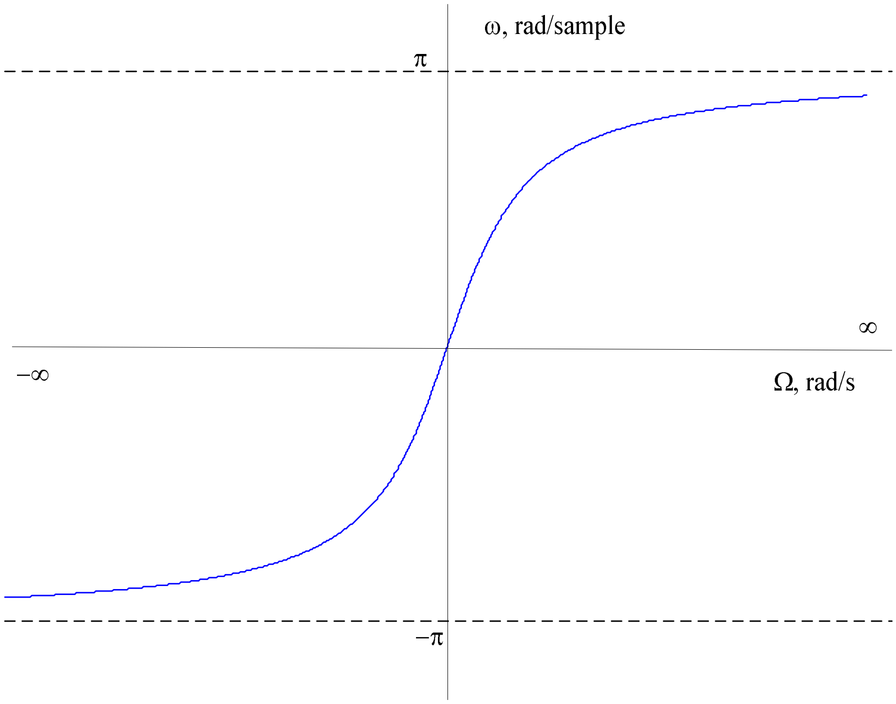
Therefore $-\left(\frac{r}{2}+\frac{1}{2r}\right)<-1 \quad \Rightarrow \quad \cos\Omega<-1$ which is impossible.

Hence if σ < 0, then | *z* | < 1 and similarly if σ > 0, then | *z* | > 1. Hence property (5.5) is satisfied.

Now, we have seen that the unit circle maps to the *j*Ω axis, therefore

$$
\begin{aligned}
s &= \frac{2}{T_s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\\
\Rightarrow j\Omega &= \frac{2}{T_s}\left(\frac{1-e^{-j\omega}}{1+z^{-j\omega}}\right)\\
&= \frac{2}{T_s}\left(\frac{1-(\cos\omega - j\sin\omega)}{1+(\cos\omega - j\sin\omega)}\right)\\
&= \frac{2}{T_s}\left(\frac{1+\cos\omega + j\sin\omega - \cos\omega - \cos^2\omega - j\sin\omega\cos\omega + j\sin\omega + j\sin\omega\cos\omega - \sin^2\omega}{(1+\cos\omega)^2+\sin^2\omega}\right)\\
&= \frac{2}{T_s}\left(\frac{2j\sin\omega}{1+\cos^2\omega + 2\cos\omega + \sin^2\omega}\right)\\
&= \frac{2}{T_s}\left(\frac{j\sin\omega}{1+\cos\omega}\right) = \frac{2}{T_s}\left(\frac{j2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}}\right)\\
\Rightarrow \Omega &= \frac{2}{T_s}\tan\tfrac{\omega}{2} \qquad (5.6)
\end{aligned}
$$

From equation (5.6) we can deduce that the relationship between the continuous time and discrete time frequencies is not linear.



Note that all frequencies in Ω are mapped between the range $-\pi$ to π for ω, but frequencies are *warped*.

To avoid problems arising with this warping of frequencies, the analogue filter characteristics are first *pre-warped* by the transformation before designing the filter in the analogue domain. So if the required cut-off frequency for the digital filter is ω_c rad/sample, then the cut-off frequency for the analogue filter, Ω_c is chosen to be

$$\Omega_c = \frac{2}{T_s} \tan \frac{\omega_c}{2} \tag{5.7}$$

If the required cut-off frequency of the digital filter is instead given as a final value in terms of rad/s (denote this by Ω_d), then

$$\begin{aligned} \Omega_c &= \frac{2}{T_s} \tan \frac{\frac{\Omega_d}{2\pi / T_s} 2\pi}{2} \\ &= \frac{2}{T_s} \tan \frac{\Omega_d T_s}{2} \end{aligned} \tag{5.8}$$

5.3.1 Design Procedure

1) From the required characteristics of the digital filter, determine Ω_d and T_s to satisfy the sampling theorem.

2) Determine Ω_c from equation (5.8).

3) Choose the appropriate $H(s)$ to meet the requirements of the digital filter (e.g. 2^{nd} order Butterworth, or 4^{th} order Chebyshev etc.)

4) Apply the bilinear transformation given by equation (5.2) on $H(s)$ to obtain $H(z)$.

5) Realise $H(z)$.